

Supplementary Materials for “Robust Covariance-Assisted Tensor Response Regression”

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This document contains the supplementary materials for the paper “Robust Covariance Assisted Tensor Response Regression”. It provides additional technical lemmas and proofs for Theorem 1 of the paper.

1 Additional Lemmas

Let $\mathbf{E}_i \sim t(0, \Sigma, \nu)$. By definition, $\mathbf{E}_i$ can be written as $1/\sqrt{G_i}\mathbf{Q}_i$, where $\mathbf{Q}_i \sim N(0, \Sigma)$.

Lemma 1 (Wang et al. (2023)). *Let $\gamma_i = 1/\sqrt{G_i}$ and $t_i = \gamma_i^2 \text{tr}(\Sigma)/p$. Then we have*

$$\mathbb{P}(|\frac{1}{p}\text{vec}(\mathbf{E}_i)^T \text{vec}(\mathbf{E}_i) - t_i| \geq \epsilon_i/2|\gamma_i) \leq 2 \exp\{-cp \min(\frac{c_1\epsilon_i^2}{\gamma_i^4}, \frac{c_2\epsilon_i}{\gamma_i^2})\}. \quad (1)$$

Let $\tilde{t}_i = \|\mathbf{Y}_i - \hat{\mathbf{B}}^{\text{OLS}} \bar{\times}_{M+1} \mathbf{X}_i\|_F/p$, which is the inverse of $\hat{\omega}_i$.

Lemma 2. *We have $\max_i |\tilde{t}_i G_i - \text{tr}(\Sigma)/p| = O(\sqrt{\log(n)/p})$ with probability at least $1 - 4n^{-C_1}$.*

Proof. Let $\mathbf{G} = \mathbf{B}_{(M+1)}^T$, and $\hat{\mathbf{G}}$ and $\hat{\mathbf{G}}^{\text{OLS}}$ be defined similarly, and $t_i = \gamma_i^2 \text{tr}(\Sigma)/p$. We first decompose $\tilde{t}_i$ as follows.

$$\begin{aligned} \tilde{t}_i &= \frac{1}{p}\text{vec}(\mathbf{E}_i)^T \text{vec}(\mathbf{E}_i) + \frac{2}{p}\mathbf{E}_i^T (\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i + \frac{1}{p}((\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i)^T (\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i \\ &\leq \frac{1}{p}\text{vec}(\mathbf{E}_i)^T \text{vec}(\mathbf{E}_i) + \frac{2}{p}\sqrt{\text{vec}(\mathbf{E}_i)^T \text{vec}(\mathbf{E}_i)}\sqrt{((\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i)^T (\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i} \\ &\quad + \frac{1}{p}((\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i)^T (\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i. \end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{P}(|\tilde{t}_i - t_i| \geq 2\epsilon) &\leq \mathbb{P}(|\frac{1}{p}\text{vec}(\mathbf{E}_i)^T \text{vec}(\mathbf{E}_i) - t_i| \geq \epsilon/2) \\
&\quad + \mathbb{P}(\frac{2}{p}\sqrt{\text{vec}(\mathbf{E}_i)^T \text{vec}(\mathbf{E}_i)}\sqrt{((\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i)^T(\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i} \geq \epsilon) \\
&\quad + \mathbb{P}(\frac{1}{p}((\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i)^T(\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i \geq \epsilon/2) \\
&\leq 2\mathbb{P}(|\frac{1}{p}\text{vec}(\mathbf{E}_i)^T \text{vec}(\mathbf{E}_i) - t_i| \geq \epsilon/2) + 2\mathbb{P}(\frac{1}{p}((\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i)^T(\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i \geq \epsilon/2).
\end{aligned}$$

Let $\epsilon_i^* = A\sqrt{\log(n)/pt_i}$, where $A > 1/(\sqrt{C_1 c_1} \text{tr}(\mathbf{\Sigma})/p)$. By Lemma 1, we have

$$n\mathbb{P}(|\frac{1}{p}\mathbf{E}_i^T \mathbf{E}_i - t_i| \geq \epsilon_i^*/2 \mid G_1, \dots, G_n) \leq 2n^{-C_1}$$

for a positive constant C_1 . By taking expectation with respect to $G_1, \dots, G_n$ on both sides of last inequality, we have

$$n\mathbb{P}(|\frac{1}{p}\mathbf{E}_i^T \mathbf{E}_i - t_i| \geq \epsilon_i/2) \leq 2n^{-C_1}.$$

Then note that

$$\sqrt{n}(\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G}) \mid (G_1, \dots, G_n) \sim \text{TN}(0, \mathbf{\Sigma}, \mathbf{\Sigma}_{\mathbf{X}, G}^{-1}).$$

It follows that

$$(\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i \mid (G_1, \dots, G_n) \sim N(0, \mathbf{X}_i^T \mathbf{\Sigma}_{\mathbf{X}, G}^{-1} \mathbf{X}_i/n \cdot \mathbf{\Sigma})$$

Then by Lemma 1, we have

$$\mathbb{P}(|\frac{1}{p}((\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i)^T(\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i - \mathbf{X}_i^T \mathbf{\Sigma}_{\mathbf{X}, G}^{-1} \mathbf{X}_i t_i/n| \geq \epsilon_i/2 \mid G_1, \dots, G_n) \leq 2 \exp\{-\tilde{c}p \min(\frac{\tilde{c}_1 \epsilon_i^2}{\gamma_i^4}, \frac{\tilde{c}_2 \epsilon_i}{\gamma_i^2})\}$$

Let $\epsilon_i^* = A\sqrt{\log(n)/pt_i}$, where $A > 1/(\sqrt{C_1 \tilde{c}_1} \text{tr}(\mathbf{\Sigma})/p)$, we have

$$n\mathbb{P}(\frac{1}{p}((\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i)^T(\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i \geq \epsilon_i^*/2 + c_3 M_x q t_i/n) \leq 2n^{-C_1}.$$

Then let $\epsilon = \epsilon_i^* + 2c_3 M_x q t_i/n$, $\mathcal{D}_i = \{|\tilde{t}_i - t_i| \leq \epsilon_i/2\}$ and $\mathcal{D} = \bigcap_i \mathcal{D}_i$. By Union bound, we have

$$\mathbb{P}(\mathcal{D}^c) \leq 2n\mathbb{P}(|\frac{1}{p}\mathbf{E}_i^T \mathbf{E}_i - t_i| \geq \epsilon_i/2) + 2n\mathbb{P}(\frac{1}{p}((\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i)^T(\hat{\mathbf{G}}^{\text{OLS}} - \mathbf{G})\mathbf{X}_i \geq \epsilon_i/2) \leq 4n^{-C_1}$$

It follows that $\max_i |\tilde{t}_i G_i - \text{tr}(\mathbf{\Sigma})/p| = O(\sqrt{\log(n)/p} + 1/n)$, with probability at least $1 - 4n^{-C_1}$ for a constant $C_1 > 0$. $\square$

A direct conclusion follows from Lemma 2 is $|\hat{\omega}_i/G_i - p/\text{tr}(\mathbf{\Sigma})| = O(\sqrt{\log(n)/p} + 1/n)$ with probability at least $1 - 4n^{-C_1}$.

Lemma 3. Let $\mathbf{Z}_i \in \mathbb{R}^{p_1 \times \dots \times p_M}$, $i = 1, \dots, n$, are i.i.d random variables from $\text{TN}(0, \mathbf{I}_{p_1}, \dots, \mathbf{I}_{p_M})$, $\mathbf{A} \in \mathbb{R}^{p-m \times p-m}$ be a positive and symmetric definite matrix whose eigenvalues are bounded above by a constant c , and α_i , $i = 1, \dots, n$, are positive constants. Then for any fixed vector $\mathbf{x} \in \mathcal{S}^{p-m-1} = \{\mathbf{x} : \mathbf{x} \in \mathbb{R}^{p-m}, \|\mathbf{x}\| = 1\}$ and any $\epsilon \geq 0$, we have

$$\begin{aligned} \mathbb{P}(|\frac{1}{np-m} \sum_{i=1}^n \alpha_i \mathbf{x}^T (\mathbf{Z}_i)_{(m)} \mathbf{A} (\mathbf{Z}_i)_{(m)}^T \mathbf{x} - \frac{\text{tr}(\mathbf{A}) \sum_{i=1}^n \alpha_i}{np-m}| \geq \epsilon) \\ \leq 2 \exp\{-Cnp-m \min(\frac{\epsilon^2}{16c^2 \sum_{i=1}^n \alpha_i^2/n}, \frac{\epsilon}{4c \max_i \alpha_i})\}. \end{aligned}$$

Proof. Let $\mathbf{P} \mathbf{\Delta} \mathbf{P}^T$ be the eigenvalue decomposition of $\mathbf{A}$, where $\mathbf{\Delta}$ is a $p-m \times p-m$ diagonal matrix whose i -th diagonal elements is δ_i . Let $(\mathbf{W}_i)_{(m)} = (\mathbf{Z}_i)_{(m)} \mathbf{P}$. We know that $\text{vec}((\mathbf{W}_i)_{(m)}) \sim N(0, \mathbf{I}_p)$. Then we have

$$\begin{aligned} \frac{1}{np-m} \sum_{i=1}^n \alpha_i \mathbf{x}^T (\mathbf{Z}_i)_{(m)} \mathbf{A} (\mathbf{Z}_i)_{(m)}^T \mathbf{x} &= \frac{1}{np-m} \sum_{i=1}^n \alpha_i \mathbf{x}^T (\mathbf{W}_i)_{(m)} \mathbf{\Delta} (\mathbf{W}_i)_{(m)}^T \mathbf{x} \\ &= \frac{1}{np-m} \sum_{i=1}^n \sum_{l=1}^{p-m} \alpha_i \delta_l \mathbf{x}^T (\mathbf{W}_i)_{(m),l} (\mathbf{W}_i)_{(m),l}^T \\ &= \frac{1}{np-m} \sum_{i=1}^n \sum_{l=1}^{p-m} \alpha_i \delta_l \mathbf{x}^T (\mathbf{W}_i)_{(m),l} (\mathbf{W}_i)_{(m),l}^T \mathbf{x}, \end{aligned}$$

where $\mathbf{W}_{(m),l}$ is the l -th column of $\mathbf{W}_{(m)}$. Note that $\mathbf{x}^T (\mathbf{W}_i)_{(m),l} \sim N(0, 1)$ and $\mathbf{x}^T (\mathbf{W}_i)_{(m),l}$ are independent for $l = 1, \dots, p-m$ and $i = 1, \dots, n$.

By Bernstein's inequality, we have

$$\begin{aligned} \mathbb{P}(|\frac{1}{np-m} \sum_{i=1}^n \sum_{l=1}^{p-m} \alpha_i \delta_l \mathbf{x}^T (\mathbf{W}_i)_{(m),l} (\mathbf{W}_i)_{(m),l}^T \mathbf{x} - \frac{1}{p-m} \text{tr}(\mathbf{A})| \leq \epsilon) \\ \leq 2 \exp\{-Cp-m \min(\frac{\epsilon^2}{16 \sum_{i=1}^n \sum_{l=1}^{p-m} \alpha_i^2 \delta_l^2 / (np-m)}, \frac{\epsilon}{4 \max_l \delta_l \max_i \alpha_i})\}. \end{aligned}$$

Since $\max_l \delta_l \leq c$, we have the desired conclusion. $\square$

Lemma 4. Let $\mathbf{Z}_i \sim \text{TN}(0, \mathbf{I}_{p_1}, \dots, \mathbf{I}_{p_M})$, for $i = 1, \dots, n$, independently, and

$$\mathbf{L} = \Sigma_m^{1/2} \left\{ \frac{p}{np-m \text{tr}(\Sigma)} \sum_{i=1}^n \alpha_i (\mathbf{Z}_i)_{(m)} \left(\bigotimes_{m' \neq m} \Sigma_{m'} \right) (\mathbf{Z}_i)_{(m)}^T \right\} \Sigma_m^{1/2}.$$

We have

$$\mathbb{P}(\|\mathbf{L} - \frac{p_m \sum_{i=1}^n \alpha_i}{n \text{tr}(\mathbf{\Sigma}_m)} \mathbf{\Sigma}_m\|_2 \geq \epsilon) \leq \exp(C_1 p_m - C_2 n p_{-m} \min(\epsilon^2 / (\sum_{i=1}^n \alpha_i^2 / n), \epsilon / \max_i \alpha_i)),$$

for some constant C_1 and C_2 .

This lemma follows from Lemma 3.

Lemma 5. For $m = 1, \dots, M$, $\|\widehat{\mathbf{\Sigma}}_m - \frac{p_m}{\text{tr}(\mathbf{\Sigma}_m)} \mathbf{\Sigma}_m\|_2 = O(\sqrt{\frac{p_m}{np_{-m}}} + \sqrt{\log(n)/p} + \sqrt{1/n})$ with probability at least $1 - C_1 n^{-C_2} - C_3 \exp(-p_m)$.

Proof. We decompose $\widehat{\mathbf{\Sigma}}_m$ in the following ways:

$$\begin{aligned} \widehat{\mathbf{\Sigma}}_m &= \frac{1}{np_{-m}} \sum_{i=1}^n \widehat{\omega}_i (\mathbf{Y}_i - \mathbf{B} \bar{\mathbf{x}}_{(M+1)} \mathbf{X}_i)_{(m)} (\mathbf{Y}_i - \mathbf{B} \bar{\mathbf{x}}_{(M+1)} \mathbf{X}_i)_{(m)}^T \\ &\quad + \frac{1}{np_{-m}} \sum_{i=1}^n \widehat{\omega}_i (\mathbf{Y}_i - \mathbf{B} \bar{\mathbf{x}}_{(M+1)} \mathbf{X}_i)_{(m)} \{(\mathbf{B} - \widehat{\mathbf{B}}^{\text{OLS}}) \bar{\mathbf{x}}_{(M+1)} \mathbf{X}_i\}_{(m)}^T \\ &\quad + \frac{1}{np_{-m}} \sum_{i=1}^n \widehat{\omega}_i \{(\mathbf{B} - \widehat{\mathbf{B}}^{\text{OLS}}) \bar{\mathbf{x}}_{(M+1)} \mathbf{X}_i\}_{(m)} (\mathbf{Y}_i - \mathbf{B} \bar{\mathbf{x}}_{(M+1)} \mathbf{X}_i)_{(m)}^T \\ &\quad + \frac{1}{np_{-m}} \sum_{i=1}^n \widehat{\omega}_i \{(\mathbf{B} - \widehat{\mathbf{B}}^{\text{OLS}}) \bar{\mathbf{x}}_{(M+1)} \mathbf{X}_i\}_{(m)} \{(\mathbf{B} - \widehat{\mathbf{B}}^{\text{OLS}}) \bar{\mathbf{x}}_{(M+1)} \mathbf{X}_i\}_{(m)}^T \\ &= \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 + \mathbf{L}_4. \end{aligned}$$

Note that $\mathbf{Y}_i - \mathbf{B} \bar{\mathbf{x}}_{(M+1)} \mathbf{X}_i = \mathbf{E}_i = \frac{1}{\sqrt{G_i}} \mathbf{\Sigma}^{1/2} \mathbf{Z}_i$, where $\mathbf{Z}_i \sim \text{TN}(0, \mathbf{I}_{p_1}, \dots, \mathbf{I}_{p_M})$. We can further decompose $\mathbf{L}_1$ in the following ways:

$$\begin{aligned} \mathbf{L}_1 &= \mathbf{\Sigma}_m^{1/2} \left\{ \frac{1}{np_{-m}} \sum_{i=1}^n \widehat{\omega}_i / G_i (\mathbf{Z}_i)_{(m)} \left(\bigotimes_{m' \neq m} \mathbf{\Sigma}_{m'} \right) (\mathbf{Z}_i)_{(m)}^T \right\} \mathbf{\Sigma}_m^{1/2} \\ &= \mathbf{\Sigma}_m^{1/2} \left\{ \frac{p}{np_{-m} \text{tr}(\mathbf{\Sigma})} \sum_{i=1}^n (\mathbf{Z}_i)_{(m)} \left(\bigotimes_{m' \neq m} \mathbf{\Sigma}_{m'} \right) (\mathbf{Z}_i)_{(m)}^T \right\} \mathbf{\Sigma}_m^{1/2} \\ &\quad + \mathbf{\Sigma}_m^{1/2} \frac{1}{np_{-m}} \sum_{i=1}^n \left\{ \left(\widehat{\omega}_i / G_i - \frac{p}{\text{tr}(\mathbf{\Sigma})} \right) (\mathbf{Z}_i)_{(m)} \left(\bigotimes_{m' \neq m} \mathbf{\Sigma}_{m'} \right) (\mathbf{Z}_i)_{(m)}^T \right\} \mathbf{\Sigma}_m^{1/2} \\ &= \mathbf{L}_{11} + \mathbf{L}_{12}. \end{aligned}$$

By Lemma 4, we know that

$$\mathbb{P}(\|\mathbf{L}_{11} - \frac{p_m}{\text{tr}(\mathbf{\Sigma}_m)} \mathbf{\Sigma}_m\|_2 \geq \epsilon) \leq \exp(C_1 p_m - C_2 n p_{-m} \epsilon^2),$$

Let $\epsilon^2 = (C_1 + 1)p_m/(np_{-m}C_2)$, we have

$$\mathbb{P}(\|\mathbf{L}_{11} - \frac{p_m}{\text{tr}(\boldsymbol{\Sigma}_m)}\boldsymbol{\Sigma}_m\|_2 \geq \epsilon) \leq \exp(-p_m). \quad (2)$$

For $\mathbf{L}_{12}$, we have

$$\begin{aligned} \|\mathbf{L}_{12}\|_2 &\leq \max_i |\tilde{\omega}_i/G_i - \frac{p}{\text{tr}(\boldsymbol{\Sigma})}| \|\boldsymbol{\Sigma}_m^{1/2} \{ \frac{1}{np_{-m}} \sum_{i=1}^n (\mathbf{Z}_i)_{(m)} (\bigotimes_{m' \neq m} \boldsymbol{\Sigma}_{m'}) (\mathbf{Z}_i)_{(m)}^T \} \boldsymbol{\Sigma}_m^{1/2}\| \\ &= \frac{\text{tr}(\boldsymbol{\Sigma})}{p} \max_i |\tilde{\omega}_i/G_i - \frac{p}{\text{tr}(\boldsymbol{\Sigma})}| \|\mathbf{L}_{11}\|_2 \end{aligned}$$

Then we know that

$$\begin{aligned} \|\mathbf{L}_1 - \frac{p_m}{\text{tr}(\boldsymbol{\Sigma}_m)}\boldsymbol{\Sigma}_m\|_2 &\leq \|\mathbf{L}_{11} - \frac{p_m}{\text{tr}(\boldsymbol{\Sigma}_m)}\boldsymbol{\Sigma}_m\|_2 + \|\mathbf{L}_{12}\|_2 \leq \|\mathbf{L}_{11} - \frac{p_m}{\text{tr}(\boldsymbol{\Sigma}_m)}\boldsymbol{\Sigma}_m\|_2 \\ &\quad + \frac{\text{tr}(\boldsymbol{\Sigma})}{p} \max_i |\tilde{\omega}_i/G_i - \frac{p}{\text{tr}(\boldsymbol{\Sigma})}| (\|\mathbf{L}_{11} - \frac{p_m}{\text{tr}(\boldsymbol{\Sigma}_m)}\boldsymbol{\Sigma}_m\|_2 + \|\frac{p_m}{\text{tr}(\boldsymbol{\Sigma}_m)}\boldsymbol{\Sigma}_m\|_2) \end{aligned}$$

By Lemma 2,

$$\|\mathbf{L}_1 - \frac{p_m}{\text{tr}(\boldsymbol{\Sigma}_m)}\boldsymbol{\Sigma}_m\|_2 = O(\sqrt{p_m/(np_{-m})}) + O(\sqrt{\log(n)/p} + 1/n),$$

with probability at least $1 - 4n^{-C_1} - \exp(-p_m)$.

Next, we consider the term $\mathbf{L}_4$. Note that

$$\begin{aligned} \mathbf{L}_4 &= \frac{1}{np_{-m}} \sum_{i=1}^n \hat{\omega}_i \{(\mathbf{B} - \hat{\mathbf{B}}^{\text{OLS}}) \bar{\times}_{(M+1)} \mathbf{X}_i\}_{(m)} \{(\mathbf{B} - \hat{\mathbf{B}}^{\text{OLS}}) \bar{\times}_{(M+1)} \mathbf{X}_i\}_{(m)}^T \\ &= \frac{p}{np_{-m} \text{tr}(\boldsymbol{\Sigma})} \sum_{i=1}^n G_i \{(\mathbf{B} - \hat{\mathbf{B}}^{\text{OLS}}) \bar{\times}_{(M+1)} \mathbf{X}_i\}_{(m)} \{(\mathbf{B} - \hat{\mathbf{B}}^{\text{OLS}}) \bar{\times}_{(M+1)} \mathbf{X}_i\}_{(m)}^T \\ &\quad + \frac{1}{np_{-m}} \sum_{i=1}^n (\hat{\omega}_i/G_i - \frac{p}{\text{tr}(\boldsymbol{\Sigma})}) G_i \{(\mathbf{B} - \hat{\mathbf{B}}^{\text{OLS}}) \bar{\times}_{(M+1)} \mathbf{X}_i\}_{(m)} \{(\mathbf{B} - \hat{\mathbf{B}}^{\text{OLS}}) \bar{\times}_{(M+1)} \mathbf{X}_i\}_{(m)}^T \\ &= \mathbf{L}_{41} + \mathbf{L}_{42} \end{aligned}$$

Becasue

$$\sqrt{n}(\hat{\mathbf{B}}^{\text{OLS}} - \mathbf{B}) \mid (G_1, \dots, G_n) \sim \text{TN}(0, \boldsymbol{\Sigma}_1, \dots, \boldsymbol{\Sigma}_M, \boldsymbol{\Sigma}_{\mathbf{X}, G}^{-1}).$$

By Lemma 4, we have

$$\begin{aligned} \mathbb{P}(\|\mathbf{L}_{41} - \frac{p_m \sum_{i=1}^n \mathbf{X}_i^T \boldsymbol{\Sigma}_{\mathbf{X}, G} \mathbf{X}_i G_i / n}{n \text{tr}(\boldsymbol{\Sigma}_m)} \boldsymbol{\Sigma}_m\|_2 \geq \epsilon \mid G_1, \dots, G_n) \\ \leq \exp \left\{ C_1 p_m - C_2 n^2 p_{-m} \min \left(\frac{\epsilon^2}{\sum_{i=1}^n (G_i \mathbf{X}_i^T \boldsymbol{\Sigma}_{\mathbf{X}, G}^{-1} \mathbf{X}_i)^2 / n}, \frac{\epsilon}{\max_i (G_i \mathbf{X}_i^T \boldsymbol{\Sigma}_{\mathbf{X}, G}^{-1} \mathbf{X}_i)} \right) \right\}. \end{aligned}$$

Note that $\mathbf{X}_i$ are bounded, the eigenvalue of $\Sigma_{\mathbf{X},G}$ are lower bounded by C_3 , $\mathbf{G}_i \sim \chi_\nu^2/\nu$, which implies $\sum_{i=1}^n G_i^2/n$ are upper bounded by some constant and $\max_i G_i$ are upper bounded by $\text{clog}(n)$ with high probability. Here we let $\epsilon^2 = (C_1 + 1)p_m/(C_2 n^2 p_{-m})$. We have

$$\|\mathbf{L}_{41}\|_2 = 1/n \cdot O(1 + \sqrt{\frac{p_m}{np_{-m}}})$$

with probability at least $1 - \exp(-p_m)$. For $\mathbf{L}_{42}$, we have

$$\begin{aligned} \|\mathbf{L}_{42}\|_2 &\leq \max_i |\tilde{\omega}_i/G_i - \frac{p}{\text{tr}(\Sigma)}| \left\| \sum_{i=1}^n G_i \{(\mathbf{B} - \hat{\mathbf{B}}^{\text{OLS}}) \bar{\times}_{(M+1)} \mathbf{X}_i\}_{(m)} \{(\mathbf{B} - \hat{\mathbf{B}}^{\text{OLS}}) \bar{\times}_{(M+1)} \mathbf{X}_i\}_{(m)}^T \right\| \\ &= \frac{\text{tr}(\Sigma)}{p} \max_i |\tilde{\omega}_i/G_i - \frac{p}{\text{tr}(\Sigma)}| \|\mathbf{L}_{41}\|_2 \end{aligned}$$

Thus,

$$\|\mathbf{L}_4\|_2 = O(1/n + \sqrt{\frac{p_m}{np_{-m}}} + \sqrt{\log(n)/p})$$

with probability at least $1 - 4n^{-C_1} - \exp(-p_m)$.

Next, we consider $\mathbf{L}_2$ and $\mathbf{L}_3$. Let $\mathbb{X} \in \mathbb{R}^{q \times n}$ be the stacked sample matrix of $\mathbf{X}_i$, $\mathbb{E} \in \mathbb{R}^{p_1 \times \dots \times p_M \times n}$ be the stacked sample tensor of $\mathbf{E}_i$, and $\mathbb{W} \in \mathbb{R}^{n \times n}$ be a diagonal matrix with the i -th diagonal element to be $\hat{\omega}_i$. We have

$$\begin{aligned} \|\mathbf{L}_2\|_2 &= \left\| \frac{1}{np_{-m}} \{(\hat{\mathbf{B}}^{\text{OLS}} - \mathbf{B}) \times_{M+1} \mathbb{X} \mathbb{W}^{1/2}\}_{(m)} \{\mathbb{E} \times_{M+1} \mathbb{W}^{1/2}\}_{(m)}^T \right\|_2 \\ &\leq \frac{1}{np_{-m}} \left\| \{(\hat{\mathbf{B}}^{\text{OLS}} - \mathbf{B}) \times_{M+1} \mathbb{X} \mathbb{W}^{1/2}\}_{(m)} \{(\hat{\mathbf{B}}^{\text{OLS}} - \mathbf{B}) \times_{M+1} \mathbb{X} \mathbb{W}^{1/2}\}_{(m)}^T \right\|_2^{1/2} \\ &\quad \cdot \left\| \{\mathbb{E} \times_{M+1} \mathbb{W}^{1/2}\}_{(m)} \{\mathbb{E} \times_{M+1} \mathbb{W}^{1/2}\}_{(m)}^T \right\|_2^{1/2} \\ &\leq \frac{1}{np_{-m}} \left\| \sum_{i=1}^n \hat{\omega}_i \{(\hat{\mathbf{B}}^{\text{OLS}} - \mathbf{B}) \bar{\times}_{M+1} \mathbf{X}_i\}_{(m)} \{(\hat{\mathbf{B}}^{\text{OLS}} - \mathbf{B}) \bar{\times}_{M+1} \mathbf{X}_i\}_{(m)}^T \right\|_2^{1/2} \left\| \sum_{i=1}^n \hat{\omega}_i (\mathbf{E}_i)_{(m)} (\mathbf{E}_i)_{(m)}^T \right\|_2^{1/2} \\ &= \sqrt{\|\mathbf{L}_1\|_2 \|\mathbf{L}_4\|_2}. \end{aligned}$$

It follows that

$$\|\mathbf{L}_2\| = O(\sqrt{1/n} + \sqrt{\frac{p_m}{np_{-m}}} + \sqrt{\log(n)/p})$$

with probability at least $1 - 8n^{-C_1} - 2\exp(-p_m)$. Note that $\mathbf{L}_3 = \mathbf{L}_2^T$, we have the same conclusion for $\mathbf{L}_3$.

Lemma 6. *The estimated envelope score satisfies that $|\hat{\phi}_l - \phi_l| = O(\max_m \sqrt{p_m/(np_{-m})} + \sqrt{1/n} + \sqrt{\log(n)/p})$ with probability at least $1 - 4n^{-C_1} - C_2 \exp(-C_3 C_M) - C_4 \sum_{m=1}^M \exp(-p_m)$, for all $l = 1, \dots, p$.*

Proof. For any fixed l ,

$$|\hat{\phi}_l - \phi_l| = |||[\hat{\mathbf{B}}; \hat{\mathbf{v}}_{l_1}^{(1)}, \dots, \hat{\mathbf{v}}_{l_M}^{(M)}]||_2 - ||[\mathbf{B}; \mathbf{v}_{l_1}^{(1)}, \dots, \mathbf{v}_{l_M}^{(M)}]||_2|,$$

for some $(l_1, \dots, l_M)$. Note that

$$\begin{aligned} & |||[\hat{\mathbf{B}}; \hat{\mathbf{v}}_{l_1}^{(1)}, \dots, \hat{\mathbf{v}}_{l_M}^{(M)}]||_2 - ||[\mathbf{B}; \mathbf{v}_{l_1}^{(1)}, \dots, \mathbf{v}_{l_M}^{(M)}]||_2| \\ & \leq |||[\hat{\mathbf{B}}; \hat{\mathbf{v}}_{l_1}^{(1)}, \dots, \hat{\mathbf{v}}_{l_M}^{(M)}] - [\mathbf{B}; \hat{\mathbf{v}}_{l_1}^{(1)}, \dots, \hat{\mathbf{v}}_{l_M}^{(M)}]||_2 + ||[\mathbf{B}; \hat{\mathbf{v}}_{l_1}^{(1)}, \dots, \hat{\mathbf{v}}_{l_M}^{(M)}] - [\mathbf{B}; \mathbf{v}_{l_1}^{(1)}, \dots, \mathbf{v}_{l_M}^{(M)}]||_2 \\ & \leq ||[\hat{\mathbf{B}} - \mathbf{B}; \hat{\mathbf{v}}_{l_1}^{(1)}, \dots, \hat{\mathbf{v}}_{l_M}^{(M)}]||_2 + ||(\otimes_{m=M}^1 \hat{\mathbf{v}}_{l_m}^{(m)} - \otimes_{m=M}^1 \mathbf{v}_{l_m}^{(m)})^T \mathbf{B}_{(M+1)}^T||_2 \\ & = ||(\otimes_{m=M}^1 \hat{\mathbf{v}}_{l_m}^{(m)})^T (\hat{\mathbf{B}} - \mathbf{B})_{(M+1)}^T||_2 + ||(\otimes_{m=M}^1 \hat{\mathbf{v}}_{l_m}^{(m)} - \otimes_{m=M}^1 \mathbf{v}_{l_m}^{(m)})^T \mathbf{B}_{(M+1)}^T||_2 \\ & = I + II. \end{aligned}$$

We first consider term II . By Theorem 2 in Yu et al. (2015), we know that

$$\|\sin \Theta(\mathbf{v}_{l_m}, \hat{\mathbf{v}}_{l_m})\|_F \leq \frac{2\|\hat{\Sigma}_m - \Sigma_m\|_2}{\Delta},$$

for a positive constant Δ . It follows that

$$\|\mathbf{P}_{\hat{\mathbf{v}}_{l_m}} - \mathbf{P}_{\mathbf{v}_{l_m}}\|_F \leq \frac{2\sqrt{2}\|\hat{\Sigma}_m - \Sigma_m\|_2}{\Delta}. \quad (3)$$

Then note that

$$\begin{aligned} & \|\mathbf{P}_{\otimes_{m=M}^1 \hat{\mathbf{v}}_{l_m}^{(m)}} - \mathbf{P}_{\otimes_{m=M}^1 \mathbf{v}_{l_m}^{(m)}}\|_F \\ & \leq \|(\otimes_{m=M}^2 \mathbf{P}_{\hat{\mathbf{v}}_{l_m}}) \otimes (\mathbf{P}_{\hat{\mathbf{v}}_{l_1}} - \mathbf{P}_{\mathbf{v}_{l_1}})\|_F + \sum_{k=2}^{M-1} \|(\otimes_{m=M}^{k+1} (\mathbf{P}_{\hat{\mathbf{v}}_{l_m}} - \mathbf{P}_{\mathbf{v}_{l_m}})) (\otimes_{m=j-1}^1 \mathbf{P}_{\mathbf{v}_{l_m}})\|_F \\ & \quad + \|(\mathbf{P}_{\hat{\mathbf{v}}_{l_M}} - \mathbf{P}_{\mathbf{v}_{l_M}}) \otimes (\otimes_{m=M-1}^1 \mathbf{P}_{\mathbf{v}_{l_m}})\|_F. \end{aligned}$$

By Lemma 5, we have

$$\|\mathbf{P}_{\otimes_{m=M}^1 \hat{\mathbf{v}}_{l_m}^{(m)}} - \mathbf{P}_{\otimes_{m=M}^1 \mathbf{v}_{l_m}^{(m)}}\|_F = O(\max_m \sqrt{\frac{p_m}{np_{-m}}} + \sqrt{\log(n)/p} + \sqrt{1/n})$$

with probability at least $1 - MC_1 n^{-C_2} - C_3 \sum_{m=1}^M \exp(-p_m)$.

Hence

$$II \leq \|\mathbf{P}_{\otimes_{m=M}^1 \hat{\mathbf{v}}_{l_m}^{(m)}} - \mathbf{P}_{\otimes_{m=M}^1 \mathbf{v}_{l_m}^{(m)}}\|_F \|\mathbf{B}_{(M+1)}\|_2 = O(\max_m \sqrt{\frac{p_m}{np_{-m}}} + \sqrt{\log(n)/p} + \sqrt{1/n})$$

Then we consider term I . Recall that

$$\hat{\mathbf{B}}_{(M+1)}^T = (\frac{1}{n} \sum_{i=1}^n \hat{\omega}_i \text{vec}(\mathbf{Y}_i) \mathbf{X}_i^T) (\frac{1}{n} \sum_{i=1}^n \hat{\omega}_i \mathbf{X}_i \mathbf{X}_i^T)^{-1}.$$

Then we have

$$\begin{aligned} & \|(\otimes_{m=M}^1 \widehat{\mathbf{v}}_{l_m}^{(m)})^T (\widehat{\mathbf{B}} - \mathbf{B})_{(M+1)}^T\|_2 \\ & \leq \left\| \frac{1}{n} \sum_{i=1}^n \widehat{\omega}_i (\otimes_{m=M}^1 \widehat{\mathbf{v}}_{l_m}^{(m)})^T (\text{vec}(\mathbf{Y}_i) - \mathbf{B}_{(M+1)}^T \mathbf{X}_i) \mathbf{X}_i^T \right\|_2 \left\| \left(\frac{1}{n} \sum_{i=1}^n \widehat{\omega}_i \mathbf{X}_i \mathbf{X}_i^T \right)^{-1} \right\|_2^{-1} \end{aligned}$$

Firstly, note that

$$\begin{aligned} \left\| \frac{1}{n} \sum_{i=1}^n \widehat{\omega}_i \mathbf{X}_i \mathbf{X}_i^T \right\|_2 & \leq \left\| \frac{1}{n} \sum_{i=1}^n (\widehat{\omega}_i / G_i - p / \text{tr}(\boldsymbol{\Sigma})) G_i \mathbf{X}_i \mathbf{X}_i^T \right\|_2 + \left\| \frac{p}{n \text{tr}(\boldsymbol{\Sigma})} \sum_{i=1}^n G_i \mathbf{X}_i \mathbf{X}_i^T \right\|_2 \\ & \leq (-\max_i |(\widehat{\omega}_i / G_i - p / \text{tr}(\boldsymbol{\Sigma}))| + p / \text{tr}(\boldsymbol{\Sigma})) \left\| \frac{1}{n} \sum_{i=1}^n G_i \mathbf{X}_i \mathbf{X}_i^T \right\|_2 \end{aligned}$$

By Lemma 2, the smallest eigenvalue of $\left\| \frac{1}{n} \sum_{i=1}^n \widehat{\omega}_i \mathbf{X}_i \mathbf{X}_i^T \right\|_2$ is lower bounded by a positive constant with probability at least $1 - 4n^{-C_2}$.

Then note that

$$\begin{aligned} & \left\| \frac{1}{n} \sum_{i=1}^n \widehat{\omega}_i (\otimes_{m=M}^1 \widehat{\mathbf{v}}_{l_m}^{(m)})^T (\text{vec}(\mathbf{Y}_i) - \mathbf{B}_{(M+1)}^T \mathbf{X}_i) \mathbf{X}_i^T \right\|_2 \\ & \leq \left\| \frac{1}{n} \sum_{i=1}^n (\widehat{\omega}_i / G_i - p / \text{tr}(\boldsymbol{\Sigma})) G_i (\otimes_{m=M}^1 \widehat{\mathbf{v}}_{l_m}^{(m)})^T (\text{vec}(\mathbf{Y}_i) - \mathbf{B}_{(M+1)}^T \mathbf{X}_i) \mathbf{X}_i^T \right\|_2 + \\ & \left\| \frac{p}{n \text{tr}(\boldsymbol{\Sigma})} \sum_{i=1}^n G_i (\otimes_{m=M}^1 \widehat{\mathbf{v}}_{l_m}^{(m)})^T (\text{vec}(\mathbf{Y}_i) - \mathbf{B}_{(M+1)}^T \mathbf{X}_i) \mathbf{X}_i^T \right\|_2 \\ & = I_1 + I_2. \end{aligned}$$

For I_2 , we have

$$\mathbb{P}(I_2 \geq \epsilon) = \mathbb{P}\left(\left\| \frac{p}{n \text{tr}(\boldsymbol{\Sigma})} \sum_{i=1}^n \sqrt{G_i} Q_i \mathbf{X}_i^T \right\|_2 \geq \epsilon\right),$$

where $Q_i \sim N(0, (\otimes_{m=M}^1 \widehat{\mathbf{v}}_{l_m}^{(m)})^T \boldsymbol{\Sigma} (\otimes_{m=M}^1 \widehat{\mathbf{v}}_{l_m}^{(m)}))$ (as a result of data splitting). Since all the elements of $\mathbf{X}_i$ are upper bounded by M_x , $\sqrt{G_i}$ is sub-Gaussian and independent of Q_i , by Bernstein's inequality, $I_2 \leq C_M \sqrt{1/n}$ with probability at least $1 - \exp(-C_1 C_M)$.

For term I_1 , we have

$$I_1 \leq \max_i |\widehat{\omega}_i / G_i - p / \text{tr}(\boldsymbol{\Sigma})| I_2.$$

By Lemma 2, we have $I_1 = O(\sqrt{\log(n)/p} + \sqrt{1/n}) \sqrt{1/n}$ with probability at least $1 - \exp(-C_1 C_M) - 4n^{-C_2}$.

To sum up, we have $|\widehat{\phi}_l - \phi_l| = O(\max_m \sqrt{p_m / (np - m)} + \sqrt{1/n} + \sqrt{\log(n)/p})$ with probability at least $1 - 4n^{-C_1} - C_2 \exp(-C_3 C_M) - C_4 \sum_{m=1}^M \exp(-p_m)$.

□

The Lemma tells that we can correctly select the subspace dimension with high probability. Let $\hat{\boldsymbol{\eta}}$ be an estimated basis matrix of $\mathcal{F}_{\Sigma}(\mathbf{B})$ obtained by the proposed algorithm in sample and $\boldsymbol{\eta}$ is a basis matrix for $\mathcal{F}_{\Sigma}(\mathbf{B})$. Combine this fact with Lemma 5 and Theorem 2 in Yu et al. (2015), we have the following result.

Lemma 7. *For positive integers C_1 , C_2 , and C_3 ,*

$$\|\mathbf{P}_{\hat{\boldsymbol{\eta}}} - \mathbf{P}_{\boldsymbol{\eta}}\|_2 = O(\sqrt{1/n} + \sqrt{\frac{p_m}{np_{-m}}} + \sqrt{\log(n)/p})$$

with probability at least $1 - MC_1 n^{-C_2} - C_3 \sum_{m=1}^M \exp(-p_m)$.

Denote j -th column of $\boldsymbol{\eta}$ as $\boldsymbol{\eta}_j = \otimes_{m=M}^1 \mathbf{v}_{mj}$, where $\mathbf{v}_{mj}$ is an eigenvector of $\mathbf{P}_{\Gamma_m} \Sigma_m \mathbf{P}_{\Gamma_m}$. By Theorem 2 in Yu et al. (2015), we know that

$$\|\sin \Theta(\mathbf{v}_{mj}, \hat{\mathbf{v}}_{mj})\|_F \leq \frac{2\|\hat{\Sigma}_m - \Sigma_m\|_2}{\Delta},$$

for a positive constant Δ . It follows that

$$\|\mathbf{P}_{\hat{\mathbf{v}}_{mj}} - \mathbf{P}_{\mathbf{v}_{mj}}\|_F \leq \frac{2\sqrt{2}\|\hat{\Sigma}_m - \Sigma_m\|_2}{\Delta}. \quad (4)$$

Then note that

$$\begin{aligned} \|\mathbf{P}_{\hat{\boldsymbol{\eta}}_j} - \mathbf{P}_{\boldsymbol{\eta}_j}\|_F &\leq \|(\otimes_{m=M}^2 \mathbf{P}_{\hat{\mathbf{v}}_{mj}}) \otimes (\mathbf{P}_{\hat{\mathbf{v}}_{1j}} - \mathbf{P}_{\mathbf{v}_{1j}})\|_F + \sum_{k=2}^{M-1} \|(\otimes_{m=M}^{k+1} (\mathbf{P}_{\hat{\mathbf{v}}_{mj}} - \mathbf{P}_{\mathbf{v}_{mj}})) (\otimes_{m=j-1}^1 \mathbf{P}_{\mathbf{v}_{mj}})\|_F \\ &\quad + \|(\mathbf{P}_{\hat{\mathbf{v}}_{Mj}} - \mathbf{P}_{\mathbf{v}_{Mj}}) \otimes (\otimes_{m=M-1}^1 \mathbf{P}_{\mathbf{v}_{mj}})\|_F. \end{aligned}$$

By Lemma 5, we have

$$\|\mathbf{P}_{\hat{\boldsymbol{\eta}}_j} - \mathbf{P}_{\boldsymbol{\eta}_j}\|_F = O(\max_m \sqrt{\frac{p_m}{np_{-m}}} + \sqrt{\log(n)/p} + \sqrt{1/n})$$

with probability at least $1 - MC_1 n^{-C_2} - C_3 \sum_{m=1}^M \exp(-p_m)$. It follows that

$$\|\mathbf{P}_{\hat{\boldsymbol{\eta}}} - \mathbf{P}_{\boldsymbol{\eta}}\|_F = O(\max_m \sqrt{\frac{p_m}{np_{-m}}} + \sqrt{\log(n)/p} + \sqrt{1/n})$$

with probability at least $1 - MC_1 n^{-C_2} - C_3 \sum_{m=1}^M \exp(-p_m)$.

□

2 Proof of Theorem 1

Proof. Recall that

$$\widehat{\mathbf{B}}^{\text{CATL}} = \sum_{j=1}^R \llbracket \widehat{\mathbf{B}}; \mathbf{v}_{l_1^{(j)}}^{(1)} (\mathbf{v}_{l_1^{(j)}}^{(1)})^T, \dots, \mathbf{v}_{l_M^{(j)}}^{(M)} (\mathbf{v}_{l_M^{(j)}}^{(M)})^T \rrbracket,$$

and $\widehat{\boldsymbol{\eta}}$ is a basis matrix of $\mathcal{F}_{\Sigma}(\mathbf{B})$, we have $(\widehat{\mathbf{B}}_{(M+1)}^{\text{CATL}})^T = \mathbf{P}_{\widehat{\boldsymbol{\eta}}} \widehat{\mathbf{B}}_{(M+1)}^T$, where

$$\widehat{\mathbf{B}}_{(M+1)}^T = \left(\frac{1}{n} \sum_{i=1}^n \widehat{\omega}_i \text{vec}(\mathbf{Y}_i) \mathbf{X}_i^T \right) \left(\frac{1}{n} \sum_{i=1}^n \widehat{\omega}_i \mathbf{X}_i \mathbf{X}_i^T \right)^{-1}.$$

By definition $\mathbf{B}_{(M+1)}^T = \mathbf{P}_{\boldsymbol{\eta}} \mathbf{B}_{(M+1)}^T$. We have

$$\begin{aligned} \|\widehat{\mathbf{B}}_{(M+1)}^{\text{CATL}} - \mathbf{B}_{(M+1)}^T\|_2 &= \|\mathbf{P}_{\widehat{\boldsymbol{\eta}}} \widehat{\mathbf{B}}_{(M+1)}^T - \mathbf{P}_{\boldsymbol{\eta}} \mathbf{B}_{(M+1)}^T\|_2 \\ &\leq \|\mathbf{P}_{\widehat{\boldsymbol{\eta}}} (\widehat{\mathbf{B}}_{(M+1)}^T - \mathbf{B}_{(M+1)}^T)\|_2 + \|(\mathbf{P}_{\widehat{\boldsymbol{\eta}}} - \mathbf{P}_{\boldsymbol{\eta}}) \mathbf{B}_{(M+1)}^T\|_2 \\ &\leq \|\widehat{\boldsymbol{\eta}}^T (\widehat{\mathbf{B}}_{(M+1)}^T - \mathbf{B}_{(M+1)}^T)\|_2 + \|\mathbf{P}_{\widehat{\boldsymbol{\eta}}} - \mathbf{P}_{\boldsymbol{\eta}}\|_2 \|\mathbf{B}_{(M+1)}^T\|_2. \end{aligned}$$

By Lemma 7, we have

$$\|\mathbf{P}_{\widehat{\boldsymbol{\eta}}} - \mathbf{P}_{\boldsymbol{\eta}}\|_F = O(\max_m \sqrt{\frac{p_m}{np_{-m}}} + \sqrt{\log(n)/p} + \sqrt{1/n})$$

with probability at least $1 - MC_1 n^{-C_2} - C_3 \sum_{m=1}^M \exp(-p_m)$. Also, since α_{rk} are all bounded, we have

$$\|\mathbf{P}_{\widehat{\boldsymbol{\eta}}} - \mathbf{P}_{\boldsymbol{\eta}}\|_2 \|\mathbf{B}_{(M+1)}^T\|_2 = O(\max_m \sqrt{\frac{p_m}{np_{-m}}} + \sqrt{\log(n)/p} + \sqrt{1/n})$$

with probability at least $1 - MC_1 n^{-C_2} - C_3 \sum_{m=1}^M \exp(-p_m)$.

For the first term of right hand side, we have

$$\|\widehat{\boldsymbol{\eta}}^T (\widehat{\mathbf{B}}_{(M+1)}^T - \mathbf{B}_{(M+1)}^T)\|_2 \leq \left\| \frac{1}{n} \sum_{i=1}^n \widehat{\omega}_i \widehat{\boldsymbol{\eta}}^T (\text{vec}(\mathbf{Y}_i) - \mathbf{B}_{(M+1)}^T \mathbf{X}_i) \mathbf{X}_i^T \right\|_2 \left\| \frac{1}{n} \sum_{i=1}^n \widehat{\omega}_i \mathbf{X}_i \mathbf{X}_i^T \right\|_2^{-1}.$$

Then note that

$$\begin{aligned} &\left\| \frac{1}{n} \sum_{i=1}^n \widehat{\omega}_i \widehat{\boldsymbol{\eta}}^T (\text{vec}(\mathbf{Y}_i) - \mathbf{B}_{(M+1)}^T \mathbf{X}_i) \mathbf{X}_i^T \right\|_2 \\ &\leq \left\| \frac{1}{n} \sum_{i=1}^n (\widehat{\omega}_i / G_i - p / \text{tr}(\Sigma)) \widehat{\boldsymbol{\eta}}^T (\text{vec}(\mathbf{Y}_i) - \mathbf{B}_{(M+1)}^T \mathbf{X}_i) \mathbf{X}_i^T \right\|_2 \\ &\quad + \left\| \frac{p}{n \text{tr}(\Sigma)} \sum_{i=1}^n G_i \widehat{\boldsymbol{\eta}}^T (\text{vec}(\mathbf{Y}_i) - \mathbf{B}_{(M+1)}^T \mathbf{X}_i) \mathbf{X}_i^T \right\|_2 \\ &= I_1 + I_2 \end{aligned}$$

For term I_2 ,

$$\mathbb{P}(I_2 \geq \epsilon) = \mathbb{P}\left(\left\|\frac{p}{n\text{tr}(\Sigma)} \sum_{i=1}^n \sqrt{G_i} Q_i \mathbf{X}_i^T\right\|_2 \geq \epsilon\right),$$

where $Q_i \sim N(0, \hat{\boldsymbol{\eta}}^T \Sigma \hat{\boldsymbol{\eta}})$ (as a result of data splitting). Since all the elements of $\mathbf{X}_i$ are upper bounded by M_x , $\sqrt{G_i}$ is sub-Gaussian and independent of $\mathbf{Q}_i$, by Bernstein's inequality, $I_2 \leq C_M \sqrt{1/n}$ with probability at least $1 - \exp(-C_1 C_M)$.

For term I_1 , we have

$$I_1 \leq \max_i |\hat{\omega}_i / G_i - p / \text{tr}(\Sigma)| I_2.$$

By Lemma 2, we have $I_1 = O(\sqrt{\log(n)/p} + \sqrt{1/n}) \sqrt{1/n}$ with probability at least $1 - \exp(-C_1 C_M) - 4n^{-C_2}$.

Finally, note that

$$\begin{aligned} \left\| \frac{1}{n} \sum_{i=1}^n \hat{\omega}_i \mathbf{X}_i \mathbf{X}_i^T \right\|_2 &\leq \left\| \frac{1}{n} \sum_{i=1}^n (\hat{\omega}_i / G_i - p / \text{tr}(\Sigma)) G_i \mathbf{X}_i \mathbf{X}_i^T \right\|_2 + \left\| \frac{p}{n\text{tr}(\Sigma)} \sum_{i=1}^n G_i \mathbf{X}_i \mathbf{X}_i^T \right\|_2 \\ &\leq \left(-\max_i |\hat{\omega}_i / G_i - p / \text{tr}(\Sigma)| + p / \text{tr}(\Sigma) \right) \left\| \frac{1}{n} \sum_{i=1}^n G_i \mathbf{X}_i \mathbf{X}_i^T \right\|_2 \end{aligned}$$

By Lemma 2, the smallest eigenvalue of $\left\| \frac{1}{n} \sum_{i=1}^n \hat{\omega}_i \mathbf{X}_i \mathbf{X}_i^T \right\|_2$ is lower bounded by a positive constant with probability at least $1 - 4n^{-C_2}$.

To sum up

$$\|\hat{\mathbf{B}}^{\text{CATL}} - \mathbf{B}\|_F = O(\sqrt{1/n} + \max_m \sqrt{\frac{p_m}{np_{-m}}} + \sqrt{\log(n)/p})$$

with probability at least $1 - 4n^{-C_1} - C_2 \exp(-C_3 C_M) - C_4 \sum_{m=1}^M \exp(-p_m)$.

□

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